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Is the renormalisation group useless in turbulence?

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ABSTRACT. – Several different ways of using the Navier-Stokes equations to derive a path integral generating functional Z for turbulence have been described in the literature. We argue that Z should be formulated so as to avoid infra-red divergence, even though this approach yields a conventional field theory which is difficult to handle, because it contains both cubic and quartic nonlinear interactions that compete on an equal footing. Notwithstanding this difficulty, application of a conventional loop expansion, together with its associated renormalisation group (RG) analysis, enables one to show that the usual 1-loop approximation can only ever yield standard K41 results, and that quantitative information about intermittency requires a much more difficult 2-loop approximation. © Elsevier, Paris

1. Introduction

Once one accepts that a statistical description of turbulence is justified, it is natural to ask the question: what is the functional probability distribution $P(\mathbf{u})$ of a particular realisation of the velocity field $\mathbf{u}$? This question is not, of course, new. It has been debated now for over four decades. Hopf (1952) was the first to tackle it. Later Edwards (1964) derived a Liouville type equation for $P(\mathbf{u})$ from the Navier-Stokes (NS) equations.

By about 1970 formally exact solutions of Liouville's equation had been found in the form of path integrals (such as that of Rosen, 1971). But this approach was not carried through to a successful conclusion, and was soon after replaced by a formal treatment of the statistical dynamics of classical field equations devised by Martin, Siggia and Rose (1973). Their approach led to a generating functional (MSR) for the velocity field correlation functions which again took the form of a path integral. But it differed from previous work in being more amenable to systematic treatment by standard methods.

The MSR generating functional continues to be widely used in the literature as a basis for turbulence calculations. However, other path integral generating functionals for turbulence derived from the NS equations are still being discussed in the literature. A recent one by Thacker (1997), for example, seeks to uncover the symplectic geometry of the phase space.

Since the generating functional Z contains all the statistical information about turbulent fluctuations that is sought, the crucial question, then, is: how can it be evaluated? In this paper, our aim is to apply the standard method of evaluating path integrals like Z, namely, the so-called loop expansion, together with its associated renormalisation group (RG) transformation. This particular approach is of interest because, in principle, it provides a means of calculating anomalous scaling exponents for structure functions. We shall attempt to explain why this is so in a manner that is intended to be accessible to readers lacking a field theory background, by offering a self-contained, albeit succinct, exposition of our approach. Inevitably, this will mean sacrificing some detail but suitable references are provided for further information. We shall indicate what progress has

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been made and seek to answer the question as to whether useful quantitative results about intermittency can be derived from the NS equations using RG methods.

The loop expansion is obtained by reorganising a standard expansion of Z in powers of the nonlinear coupling into certain independent irreducible parts, the so-called vertex functions. Their generating functional, Γ, is derived in Section 2. The loop expansions of the vertex functions can be represented as connected topological (Feynman) diagrams according to standard conventions. The rules for evaluating these diagrams are summarised in Section 3 together with the diagrams relating to the key low order vertex functions. In the context of homogeneous turbulence, the loop expansion can be regarded as an expansion in terms of the ratio of the mean dissipation rate to some peak dissipation rate, such as might occur in some filamentary vortex structure.

In most field theories, some integrands associated with vertex functions simply do not decay fast enough for the corresponding integrals to converge. This so-called ultra-violet (uv) divergence occurs as a result of the failure to describe small scale physics accurately enough. This may happen in the case of homogeneous turbulence when we set up an effective theory which can focus on inertial range scales, but which does not treat dissipation range scales accurately enough. An example is the widely discussed model in which $\mathbf{u}$ is driven by a random force $\mathbf{f}$ whose correlation function varies as a negative power of wavenumber, (see, for example, DeDomincis and Martin, 1979, Ronis, 1987, Teodorovich, 1989).

However, such divergences need not be fatal and may even be turned to advantage. The point is that, because the small scale physics has been simplified, or omitted, it becomes possible to tackle problems which otherwise would be much harder. The calculation of anomalous exponents of structure functions is a case in point.

These uv divergences can, in fact, be handled by standard renormalisation procedures. In effect, these sweep the divergences up into renormalised versions of the basic parameters, which in the present case are, of course, the viscosity ν and the dissipation rate ε. This is done through the introduction of renormalisation constants Z_ν and Z_ε . We show how these constants are evaluated in Section 4. This process compensates for the incomplete description of the small scale physics and it enables the renormalised parameters to be interpreted as the true physical values. It then becomes possible to make finite predictions for the correlation functions themselves. This approach relies on the insensitivity of universal behaviour to the short distance structure, significant though that might be.

In formulations based on MSR type generating functionals, one may also encounter divergences at small wavenumbers, the so-called infra-red (if) divergences, (Adzhemyan *et al.*, 1989, Teodorovich, 1992). These appear in association with a turbulence response function. This suggests, therefore, that they might be avoided by developing a formulation of Z in which the response function is not a primary ingredient.

One can see how to accomplish this by noting that the MSR type representations of Z involve not only the actual velocity field $\mathbf{u}$, but, in addition, an auxiliary non-physical field which mimics the Hermitian conjugate of conventional conservative complex fields. It is the latter which introduces the response function. Hence, the if problems can be avoided by expressing the generating functional solely in terms of $P(\mathbf{u})$. One then has a situation similar to that occurring in quantum electrodynamics in which if divergences cancel leaving only finite terms.

The calculation of Z in terms of $P(\mathbf{u})$ certainly makes physical sense and we shall pursue this approach below, using the expression for $P(\mathbf{u})$ derived previously, Giles (1994). But there is a substantial penalty to pay in that Z then involves both cubic and quartic nonlinear interaction terms, which compete on an equal footing, as compared to the MSR functional which only involves a cubic term. Unfortunately, this makes it exceedingly difficult to carry out the loop expansion.

It is important to appreciate that renormalisation has to be carried out relative to some arbitrarily chosen scale represented by a wavenumber μ. This can be thought of as a coarse-graining down to this scale. Because renormalisation has this arbitrary element, it has the important property of being invariant under reparametrisation. It is this invariance that we make use of in applications to turbulence. The expression of this invariance in the form of a RG equation is undertaken in Section 5, where the crucial role of the renormalisation constants will become apparent. The universal behaviour that we seek to describe by the scaling exponents emerges from the fixed points of the RG flows described by this equation, and we present the relevant solution in Section 5. Using this solution, we demonstrate that at 1-loop order one discovers nothing about how fluctuations of the dissipation might affect scaling exponents and obtains only standard K41 results. This is disappointing, but it does not necessarily imply that the RG approach is useless, adding nothing to what can be inferred from dimensional considerations. What it means is that it will only be possible to capture corrections to exponents at the 2-loop order. At the present time, however, little attention has been given to 2-loop diagrams even within the widely studied MSR framework, with the exception of a paper by Teodorovich (1992). Consequently, no general framework has yet emerged which shows how standard field theory methods might be applied systematically to calculate anomalous scaling exponents for structure functions of arbitrary order. The present approach seems to offer this possibility and the prospects for successfully completing such calculations are discussed in the final section.

2. The generating functional

Our starting point is the Navier-Stokes equations describing an incompressible fluid of unit density, velocity $\mathbf{u}$, kinematic viscosity ν_0 and pressure p, and driven by a random stirring force $\mathbf{f}$, which are

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nu_0 \nabla^2 \mathbf{u} + \mathbf{f}, \tag{2.1}$$

and

$$\mathrm{div}\mathbf{u} = 0.$$

It will be convenient to work in the Fourier domain putting

$$\mathbf{u}(\hat{x}) = \int \exp(i\hat{k} \cdot \hat{x})\mathbf{u}(\hat{k})\, D\hat{k}, \tag{2.2}$$

where $\hat{k}$ and $\hat{x}$ denote respectively, $(\omega, \mathbf{k})$ and $(t, \mathbf{x})$, with $\hat{k} \cdot \hat{x} = \omega t - \mathbf{k} \cdot \mathbf{x}$, while $D\hat{k} = d\hat{k}/(2\pi)^{d+1}$ with $d\hat{k} = d\omega d^d\mathbf{k}$, and d denotes the number of space dimensions.

The derivation of the functional probability distribution of the velocity field, $P(\mathbf{u})$, from (2.1) has been given previously, Giles (1994). The approach is to assume that $\mathbf{u}$ is driven by a random stirring force which has a functional probability distribution $P_0(\mathbf{f})$. $P(\mathbf{u})$ is then obtained by regarding the NS equations as specifying a transformation from the force field phase space to the velocity field phase space, giving

$$P(\mathbf{u}) = JP_0(\mathbf{f}(\mathbf{u})), \tag{2.3}$$

where J denotes the Jacobian of the transformation from $\mathbf{f}$ to $\mathbf{u}$ given by the NS equations.

One normally takes $P_0(\mathbf{f})$ to be a Gaussian functional. But, because $\mathbf{f}$ is introduced as an indeterminate intervening variable, whose sole purpose is to simulate the process of feeding energy from some unspecified large scale motion to small scales, to sustain the turbulence against viscous decay, one should regard the exponent of $P_0(\mathbf{f})$ as a general functional polynomial of $\mathbf{f}$, whose specific detail should be inferred from the solution, where required. In particular, the contributions from its higher order terms should be shown to be irrelevant (thereby justifying the use of the Gaussian term only) in the sense that they are suppressed by powers of an *uv* cut-off. So far, however, no thorough study of the higher order terms appears to have been conducted, and we must therefore proceed, at least *pro tem*, on the usual basis of assuming that **f** is a Gaussian random function.

This approach yields for $P(\mathbf{u})$ an effective action which is the sum of Gaussian, S_2, cubic, S_3, and quartic, S_4, functional polynomials. The Gaussian term has the form

$$S_2(\mathbf{u}) = \frac{1}{2}\int M^{(2)}_{\alpha\beta}(\hat{k},\hat{l})u_\alpha(\hat{k})u_\beta(\hat{l})D\hat{k}\,D\hat{l}, \tag{2.4}$$

where

$$M^{(2)}_{\alpha\beta}(\hat{k},\hat{l}) = (2\pi)^{d+1}\,M^{(2)}(\hat{k},\nu_0)\,P_{\alpha\beta}(\mathbf{k})\,\delta(\hat{k}+\hat{l}).$$

Here $P_{\alpha\beta}(\mathbf{k}) = \delta_{\alpha\beta} - k_\alpha k_\beta/k^2$ and

$$M^{(2)}(\hat{k},\,\nu_0) = D(k)^{-1}\,(\omega^2+\nu_0^2k^4), \tag{2.5}$$

where $D(k)$ is the autocorrelation function of the stirring force. Note that the inverse of $M^{(2)}_{\alpha\beta}$ is rendered definite by the addition of an arbitrary longitudinal term, which is not shown explicitly. The cubic and quartic terms have the forms:

$$S_3(\mathbf{u}) = \frac{1}{3!}\int D\hat{k}\,D\hat{l}\,D\hat{m}\delta(\hat{k}+\hat{l}+\hat{m})\,M^{(3)}_{\alpha\beta\gamma}(\hat{k},\hat{l},\hat{m})\,u_\alpha(\hat{k})\,u_\beta(\hat{l})u_\gamma(\hat{m}), \tag{2.6}$$

where

$$M^{(3)}_{\alpha\beta\gamma}(\hat{k},\hat{l},\hat{m}) = i(2\pi)^{d+1}\{(i\omega+\nu_0k^2)D(k)^{-1}\,P_{\alpha\beta\gamma}(\mathbf{k}) + perms.\}, \tag{2.7}$$

and

$$\begin{aligned}S_4(\mathbf{u}) = \frac{1}{4!}\int & D\hat{k}\,D\hat{l}\,D\hat{m}\,D\hat{n}\,\delta(\hat{k}+\hat{l}+\hat{m}+\hat{n})\,M^{(4)}_{\alpha\beta\gamma\delta}(\hat{k},\hat{l},\hat{m},\hat{n})\\ &\times u_\alpha(\hat{k})u_\beta(\hat{l})u_\gamma(\hat{m})u_\delta(\hat{n}),\end{aligned} \tag{2.8}$$

where

$$M^{(4)}_{\alpha\beta\gamma\delta}(\hat{k},\hat{l},\hat{m},\hat{n}) = (2\pi)^{d+1}\{D(|\mathbf{k}+\mathbf{l}|)^{-1}\,P_{\sigma\alpha\beta}(\mathbf{k}+\mathbf{l})P_{\sigma\gamma\delta}(\mathbf{k}+\mathbf{l}) + perms.\} \tag{2.9}$$

In these coefficients, $P_{\alpha\beta\gamma}(\mathbf{k}) = k_\beta P_{\alpha\beta}(\mathbf{k}) + k_\gamma P_{\alpha\beta}(\mathbf{k})$.

The Fourier transform of the 2-point correlation function $G^{(2)}_{\alpha\beta}(\hat{k})$ defined by

$$\langle u_\alpha(\hat{k})u_\beta(\hat{l})\rangle = (2\pi)^{d+1}\,\delta(\hat{k}+\hat{l})G^{(2)}_{\alpha\beta}(\hat{k}),$$

is of particular importance for the loop expansion, especially its zero order term, which, for ease of notation, we indicate simply by $G_{\alpha\beta}(\hat{k})$. Thus,

$$G_{\alpha\beta}(\hat{k}) = M^{(2)}(\hat{k})^{-1}\,P_{\alpha\beta}(\mathbf{k}). \tag{2.10}$$

In general, the transverse components of $G^{(2)}_{\alpha\beta}$ will have the same form

$$G^{(2)}_{\alpha\beta}(\hat{k}) = G^{(2)}(\hat{k})P_{\alpha\beta}(\mathbf{k}). \tag{2.11}$$

The foregoing allows one to develop an effective theory of homogenous turbulence through suitable choice of $D(k)$. The detailed form of $D(k)$ is, of course, an imponderable, on account of the artificial nature of the stirring force. But one can assume that the quadratic coefficient $M^{(2)}(\hat{k}, \nu_0)$ has a MacLaurin expansion in k which implies an expansion for $D(k)^{-1}$ of the form $D_0^{-1}k^m(1 + d_1k + d_2k^2 + ...)$. This seems reasonable as $D(k)$ can be regarded as having a singularity at $k = 0$, corresponding to the injection of energy at large wavelengths. But, as shown previously, (Giles, 1994) the d_j terms are irrelevant in the RG sense and may therefore be ignored. Moreover, it was also shown that if the coarse-grained mean dissipation rate is to remain equal to the actual mean dissipation rate ε_0, then one must set $m = 3$, and D_0 will be proportional to ε_0. Thus, we have

$$D(k) = D_0k^{-m} \propto \varepsilon_0 k^{-3}. \tag{2.12}$$

It is easy to see from the foregoing that the coupling constant is λ for the cubic term (2.6) and λ^2 for the quartic term (2.8) where

$$\lambda = (D_0/\nu_0^3)^{1/2}. \tag{2.13}$$

Furthermore, λ has a physical dimension given by $(\text{length})^{d-d_0}$, where

$$d_0 = 4 + m = 7. \tag{2.14}$$

Both coupling constants are therefore dimensionless in a space dimension d_0. Under these circumstances, one can conclude that W will be renormalisable when $d = d_0$, and we will discuss this point further in Section 4.

Replacing the arbitrary functional $P_0(\mathbf{f})$ by a Gaussian form with autocorrelation function (2.12) corresponds to the usual RG process of 'tuning' the functional to obtain an effective probability distribution that can represent a critical state (see e.g. Binney *et al.* 1992, ch. 14). This means that (2.12) is to be interpreted not as a cubic power law forcing but rather as the only relevant term of the general Gaussian form. But, as indicated, while the higher order terms of $D(k)^{-1}$ can be shown to be irrelevant, the non-Gaussian features remain to be studied. This could be a serious weakness of the approach. Indeed, Eyink (1994) has noted that in his general formulation of the problem some of the terms that appear in the Wilson transformation may be marginal and could lead to an infinity of fixed points in the RG transformation.

With $P(\mathbf{u})$ known, we can obtain the correlation functions of $\mathbf{u}$ from the path integral generating functional

$$Z(\mathbf{J}) = \int P(\mathbf{u})\exp\left\{\int \mathbf{J}(-\hat{k}) \cdot \mathbf{u}(\hat{k})D\hat{k}\right\}\mathcal{D}\mathbf{u},$$

by functional differentiation with respect to $\mathbf{J}$ in the conventional manner (see e.g. Zinn-Justin, 1996, Ch.6).

In practice, it is more convenient to work in terms of the so-called vertex functions rather than directly in terms of the correlation functions themselves. Their generating functional $\Gamma(\mathbf{u})$ is obtained from Z in the following way (see e.g. Zinn-Justin, 1996, Ch.6). We form the generating functional of connected correlation functions (cumulants) which is

$$W(\mathbf{J}) = \log Z(\mathbf{J}).$$

Then $\Gamma(\mathbf{u})$ is given by the Legendre transform of $W(\mathbf{J})$, namely

$$\Gamma(\mathbf{u}) + W(\mathbf{J}) - \int \mathbf{J}(-\hat{k}) \cdot \mathbf{u}(\hat{k}) D\hat{k} = 0, \tag{2.15}$$

where $\mathbf{u}$ is related to $\mathbf{J}$ by

$$\mathbf{u}(\hat{k}) = (2\pi)^{d+1} \frac{\delta W}{\delta \mathbf{J}(\hat{k})}.$$

It should noted that the Jacobian in (2.3) can be ignored because the terms arising from it in perturbation theory cancel identically the terms independent of D_0 arising from the effective action itself. In fact, in homogeneous turbulence these terms are identically zero as shown by Eyink (1996).

3. The loop expansion

Standard field theory rules can be used to derive the loop expansions of the vertex functions from their generating functional Γ. These rules obviate the need for laborious calculations based on the definition (2.15). They specify the terms of the loop expansion in diagrammatic form and provide a dictionary to translate the diagrams into actual mathematical expressions, (see e.g. Binney *et al.* 1992, Ch.8). The loop expansion is a saddle point evaluation of Z which is valid for small D_0, that is small ε_0. Here small means small compared with the dissipation rate associated with the saddle point flow field configuration, which one might regard as representing some filamentary vortex structures. From this point of view such an expansion does not seem unreasonable and might be expected to yield worthwhile results.

Consider first the second order vertex function $\Gamma^{(2)}_{\alpha\beta}(\hat{k}, \hat{l})$. This is actually the inverse of the 2-point correlation function. So its lowest order term (zero-loop) is given by $G_{\alpha\beta}(\hat{k})$. For its one-loop term we refer to the relevant diagrams, which are those shown in Fig. 1, (again, see e.g. Binney *et al.* 1992, Ch.8 for further detail). In these diagrams the external links represent the arguments of $\Gamma^{(2)}$. The loop represents an integration over a wavenumber $\hat{p}$, say, while its internal links represent $G_{\alpha\beta}(\hat{k})$ evaluated at the wavevectors indicated, namely $\hat{p}$ and $\hat{k} - \hat{p}$. Wavevectors flow along the links subject to conservation at vertices, with overall conservation following from the homogeneous form

$$\Gamma^{(2)}_{\alpha\beta}(\hat{k}, \hat{l}) = \Gamma^{(2)}_{\alpha\beta}(\hat{k})(2\pi)^{d+1}\delta(\hat{k} + \hat{l}).$$

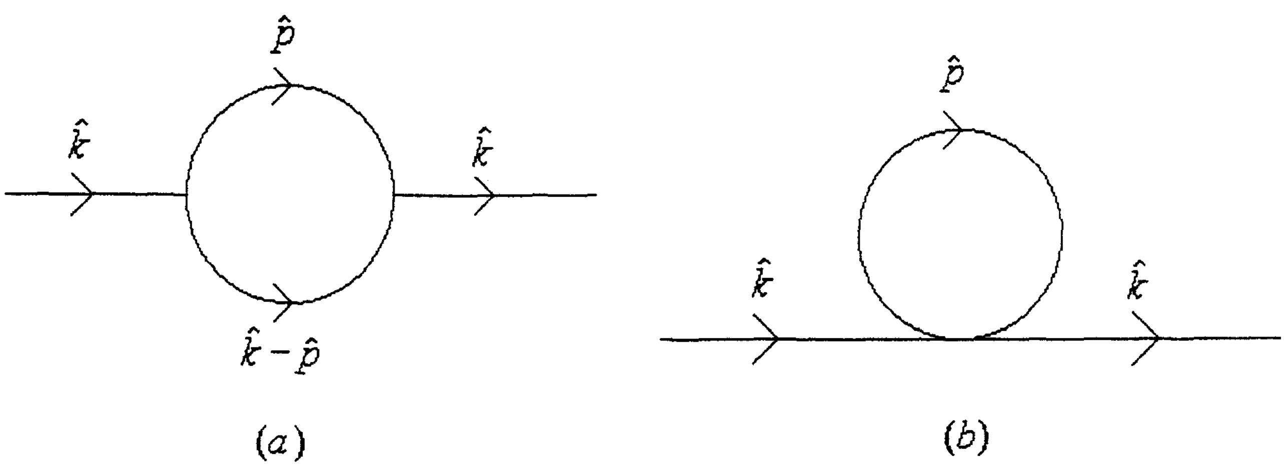

Fig. 1. – 1-loop diagrams for $\Gamma^{(2)}$.

The vertices in Figure 1(a) represents the cubic interaction coefficient (2.7), while the vertex in Figure 1(b) represents the quartic interaction coefficient (2.9). The numerical coefficients and signs to be associated with these diagrams can be determined by reference to the standard Feynman rules, (see e.g. Binney *et al.* 1992, Ch.8). Together these diagrams yield for the 1-loop term of the reduced vertex function

$$\begin{aligned}\Gamma^{(2)}_{\alpha\beta}(\hat{k})_{1\text{loop}} &= \frac{1}{2}\int D\hat{p}\, M^{(4)}_{\alpha\beta\gamma\delta}(-\hat{k},\hat{k},\hat{p},-\hat{p})G_{\gamma\delta}(\hat{p}) \\ &\quad -\frac{1}{2}\int D\hat{p}\, M^{(3)}_{\alpha\lambda\mu}(-\hat{k},\hat{p},\hat{k}-\hat{p})M^{(3)}_{\beta\sigma\nu}(\hat{k},-\hat{p},-\hat{k}+\hat{p}) \\ &\quad \times G_{\lambda\sigma}(\hat{p})G_{\mu\nu}(\hat{k}-\hat{p}).\end{aligned} \tag{3.1}$$

Consider next the 3-point vertex function $\Gamma^{(3)}_{\alpha\beta\gamma}(\hat{l},\hat{m},\hat{n})$. Its zero-loop (or tree level) term is just (2.7), while its 1-loop term is obtained from the diagrams shown in Figure 2. Figure 2(a) contributes a term

$$\begin{aligned}\int D\hat{p}\, M^{(3)}_{\sigma\lambda\mu}(-\hat{l},-\hat{p},\hat{l}+\hat{p})\, M^{(3)}_{\beta\sigma\tau}(-\hat{m},\hat{p},\hat{m}-\hat{p})\, M^{(3)}_{\gamma\rho\delta}(-\hat{n},-\hat{l},-\hat{p},\hat{p}-\hat{m}) \\ \times G_{\lambda\sigma}(\hat{p})G_{\mu\rho}(\hat{l}+\hat{p})G_{\delta\tau}(\hat{m}-\hat{p}),\end{aligned} \tag{3.2}$$

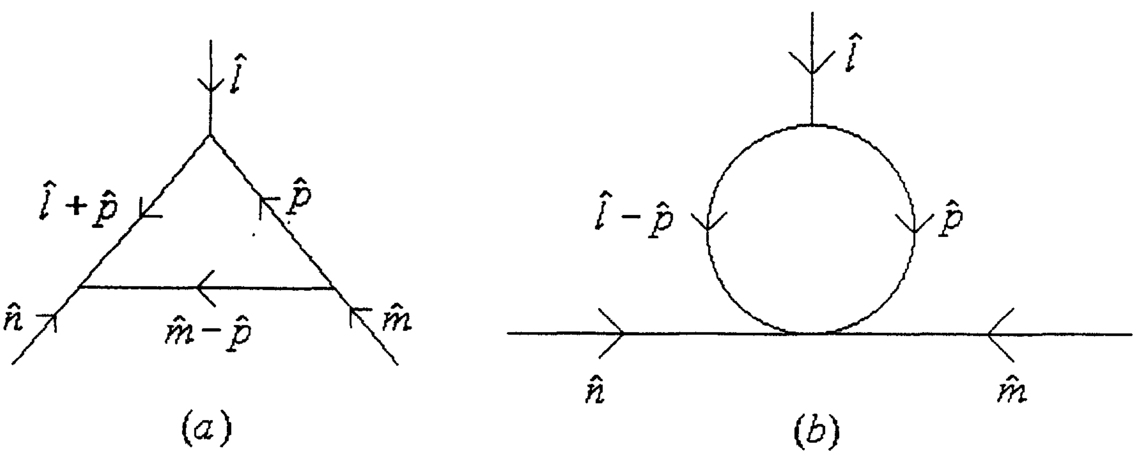

Fig. 2. – 1-loop diagrams for $\Gamma^{(3)}$.

while Figure 2 (b) contributes

$$-\frac{1}{2}\int D\hat{p}\, M^{(3)}_{\alpha\lambda\mu}(-\hat{l},\hat{p},\hat{l}-\hat{p})\, M^{(4)}_{\beta\gamma\sigma\nu}(-\hat{m},-\hat{n},-\hat{p},-\hat{l}+\hat{p})G_{\gamma\sigma}(\hat{p})\, G_{\mu\nu}(\hat{l}-\hat{p}), \tag{3.3}$$

plus its two permutations.

4. Renormalization

The idea of renormalisation theory is that the uv divergences of a field theory are to be cancelled by renormalisation of its basic parameters. To see how this works in the present case, consider the original or 'bare' quadratic term (2.4). Its parameters are the viscosity ν_0 and the mean dissipation rate ε_0. We therefore replace these quantities by renormalised values ν and ε, which are related to ν_0 and ε_0 through renormalisation constants Z_ν and Z_ε defined by

$$\nu_0 = \nu Z_\nu \qquad \text{and} \qquad \varepsilon_0 = \varepsilon Z_\varepsilon. \tag{4.1}$$

The 'bare' coefficient of (2.5), $M^{(2)}(\hat{k}, \nu_0)$, then becomes the sum of a renormalised coefficient, $M^{(2)}(\hat{k}, \nu)$, of identical form and an increment, called a counter-term, which takes the form

$$\delta M^{(2)} = (Z_1 - 1) M^{(2)}(\xi \hat{k}, \nu), \tag{4.2}$$

where

$$Z_1 = Z_\nu^2 / Z_\varepsilon.$$

Here ξ acting on $\hat{k}$ renormalises the frequency, with $\xi\hat{k} \equiv (\mathbf{k}, \xi\omega)$, where $\xi = [(Z_\varepsilon^{-1} - 1)/(Z_1 - 1)]^{1/2}$. However, with appropriate choice of vertex functions, one can determine Z_ν and Z_ε in the static limit $\omega = 0$, so that ξ will not appear below.

The freedom of choice presented by $\delta M^{(2)}$ can now be used to eliminate divergences from $\Gamma^{(2)}$. This results in finite values for ν and ε, whereas the counter-term increments

$$\Delta Z_{\nu,\varepsilon} \equiv Z_{\nu,\varepsilon} - 1$$

are adjusted to cancel the uv divergences.

In a similar way, renormalisation induces cubic and quartic counter-terms of the forms (2.6) and (2.8) with respective coefficients

$$\delta M^{(3)} = (Z_3 - 1) M^{(3)}_{\alpha\beta\gamma}(\hat{l}, \hat{m}, \hat{n}), \tag{4.3}$$

where

$$Z_3 = Z_\nu / Z_\varepsilon$$

and

$$\delta M^{(4)} = (Z_\varepsilon^{-1} - 1) M^{(4)}_{\alpha\beta\gamma\delta}(\hat{k}, \hat{l}, \hat{m}, \hat{n}). \tag{4.4}$$

Again, these counter-terms permit divergences to be eliminated from $\Gamma^{(3)}$ and $\Gamma^{(4)}$.

Renormalisable theories are characterised by the property that at least one vertex has dimension zero and no vertex has a positive dimension (see e.g. Zinn-Justin, 1996, Ch. 8). This rule indicates that W is, in fact, renormalisable in dimension $d = d_0$, because both the cubic and the quartic vertices have dimension zero when $d = d_0$ as follows from (2.13). This implies, furthermore, that the counter-terms (4.2)-(4.4) are, in fact, sufficient for the elimination of all uv divergences.

We can determine how the integrands of the 1-loop terms behave in the uv by setting $\hat{p} \equiv (\Omega, \mathbf{p}) = (xp^2, \mathbf{p})$. Then from (2.5), (2.10) and (2.12), we get $G_{\alpha\beta}(\hat{p}) \sim p^{-d_0}$, while (2.7) and (2.9) yield $M^{(3)} \sim p^{d_0 - 1}$ and $M^{(4)} \sim p^{d_0 - 2}$. Thus, in the case of $\Gamma^{(2)}$ both integrals in (3.1) diverge as p^d. This strong divergence cannot be renormalised, but, in fact, these divergent terms cancel identically and one is left with only logarithmic divergences, which can be renormalised. The same is true in the case of $\Gamma^{(3)}$. The individual integrals diverge as p^{d-1}. But detailed analysis shows that these strong divergences again cancel identically. In the case of $\Gamma^{(4)}$, the logarithmic divergences also cancel identically at 1-loop order.

A similar argument in the if shows that when the external wavenumbers are not zero the integrals behave as p^{2+d-d_0}. Hence, renormalisation at a dimension $d = d_0$ does not produce if divergent integrals. This is also true at zero wavenumbers by virtue of the cancellations of the strong singularities mentioned above.

In order to carry out the actual calculation of the renormalisation constants a means of regulating the uv divergences is required. This could be achieved through the imposition of an uv cut-off at some wavenumber $k = \Lambda$. However, we shall adopt dimensional regularisation as it is one of the most powerful regularisation procedures (see e.g. Zinn-Justin, 1996, Ch. 11). In this, expressions like (3.1)-(3.3) are analytically continued with respect to the space dimension d to arbitrary complex values. The divergences then appear as poles at $d = d_0$ and we choose the renormalisation constants so as to cancel these poles. This is the renormalisation procedure termed minimal subtraction (see e.g. Zinn-Justin 1996, Ch. 11). In general, therefore, renormalisation constants are determined at each order of the loop expansion by subtracting the singular part of the Laurent expansion of the vertex functions at the renormalisation dimension $d = d_0$.

The uv cut-off Λ remains as an underlying element. Since it may be taken as some dissipation range wavenumber less than the familiar dissipation wavenumber $k_d = (\varepsilon_0/\nu_0^3)^{1/4}$, it implies a non-dimensional coupling constant for the 'bare' theory given by $g_0 = (k_d/\Lambda)^2$ or

$$g_0 = \left(\frac{\varepsilon_0}{\nu_0^3 \Lambda^4}\right)^{1/2},$$

which is now employed instead of the dimensional coupling constant of (2.13). In the renormalised theory g_0 is replaced by a renormalised coupling constant g involving the corresponding renormalised parameters and the new scale μ that replaces Λ, giving

$$g = \left(\frac{\varepsilon}{\nu^3 \mu^4}\right)^{1/2}. \tag{4.5}$$

Consider now the 1-loop term of $\Gamma^{(2)}$ as given by (3.1). After performing the frequency integral, this expression reduces to a d-dimensional integral of the Feynman type, which can be evaluated by standard methods. An outline of the details of this calculation is given in Appendix 1, where it is shown that the singular term of $\Gamma^{(2)}_{\alpha\beta}$ is a pole with residue $P_{\alpha\beta}(\mathbf{k})(-2\alpha k^{d_0}/\nu)$ where

$$\alpha = \frac{d_0 - 1}{4(d_0 + 2)} \frac{S_d}{(2\pi)^d},$$

S_d being the surface area of a d-dimensional unit sphere. We use the quadratic counter-term and, in particular, the increment $\Delta Z_1 = Z_1 - 1$, to eliminate this pole, which yields

$$2\Delta Z_\mu - \Delta Z_\varepsilon = 2\alpha g^2/(d - d_0).$$

Similarly, we find that the 1-loop term of $\Gamma^{(3)}$, as given by (3.2) and (3.3), also has a simple pole at $d = d_0$. The choice of Z_3 which is needed to eliminate this pole can be found by considering the invariant $I_3 = l_\beta \Gamma^{(3)}_{\alpha\beta\alpha}(\hat{l}, -\hat{l}, 0)$ in the static limit. In Appendix 1, we show that the residue of I_3 is $(-\alpha k^{d_0}/\nu)i(d_0 - 1)/\nu$. We now choose the increment $\Delta Z_3 = Z_3 - 1$ so that the same invariant of the cubic counter-term cancels this pole, obtaining

$$\Delta Z_\nu - \Delta Z_\varepsilon = -\alpha g^2/(d - d_0).$$

We can now use the last two equations to determine the basic renormalisation coefficients to 1-loop order. This gives

$$\Delta Z_\nu = \alpha g^2/(d - d_0),$$

and

$$\Delta Z_\varepsilon = 0. \tag{4.6}$$

According to (4.6) it is apparent that the quartic counter-term (4.4) vanishes at 1-loop order, which implies that $\Gamma^{(4)}$ contains no divergent terms at this order. A very tedious calculation enables one to verify that all divergent terms appearing in the 1-loop expansion of $\Gamma^{(4)}$ do, indeed, cancel identically.

At 1-loop order the non-zero value of ΔZ_ν captures the effect of coarse-graining on viscosity. On the other hand the fact that ΔZ_ε is zero means that we discover nothing about how fluctuations of the dissipation rate might affect scaling exponents. In fact, at 1-loop order we only obtain standard K41 results. We can see this in more precise terms by using the invariance under reparametrisation (*i.e.* change of μ) to calculate the scaling exponent of the energy spectrum $E(k)$, which we consider next.

The minimal subtraction method has also been applied to turbulence by Ronis (1987) in an approach based on the 1-loop approximation of the MSR functional. This differs from the present approach in several respects, including the treatment of if divergence, the fact that only one renormalisation constant is taken into account (Z_ν), so that no conclusions can be reached regarding anomalous exponent, and because no use is made of the dissipation rate integral to resolve indeterminacy relating to the relevant term $D(k)^{-1}$, so that one does not obtain a definite value for ΔZ_ν.

5. Renormalisation group invariance

As we have indicated, physical quantities like the 2-point correlation function $G^{(2)}(\hat{k})$ are invariant under a reparametrisation in which $(\mu, \nu(\mu), \varepsilon(\mu))$ change to $(\mu', \nu(\mu'), \varepsilon(\mu'))$ and this invariance can be used to infer scaling exponents. The invariance is usually expressed in differential form by considering only a small change in $\mu : \mu' = \mu + \delta\mu$. The resulting RG equation can be derived as follows in the case of $G^{(2)}(\hat{k})$, (see e.g. DeDominicis and Martin, 1979).

Dimensional considerations indicate that the 2-point correlation function $G^{(2)}(\hat{k})$ of (2.11) has the form

$$G^{(2)}(\hat{k}) = \frac{\varepsilon_0}{\nu_0^2} G_0(\mathbf{k}, \omega/\nu_0, g_0, \Lambda),$$

with a corresponding renormalised form

$$G^{(2)}(\hat{k}) = \frac{\varepsilon}{\nu^2} G_0(\mathbf{k}, \omega/\nu_0, g, \Lambda).$$

Invariance under reparametrisation implies $dG^{(2)}/d\mu = 0$, as is apparent from the first form. Therefore, differentiation of the second form yields, after taking (4.1) into account,

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + \eta_\nu\omega\frac{\partial}{\partial\omega}\right) G_R^{(2)} + (2\eta_\nu - \eta_\varepsilon)\, G_R^{(2)} = 0, \tag{5.1}$$

where

$$\beta(g) = \mu \frac{\partial g}{\partial \mu},$$
$$\eta_\nu = \mu \frac{\partial}{\partial \mu} \log Z_\nu,$$

and

(5.2) $$\eta_\varepsilon = \mu \frac{\partial}{\partial \mu} \log Z_\varepsilon.$$

The RG coefficients β, η_ν and η_ε are readily calculated from the counter-terms, as demonstrated in the previous section. With minimal subtraction they appear as functions of μ only, which permits (5.1) to be solved readily using the method of characteristics. Particular interest here attaches to the solution at a zero $g = g_\bullet$ of β with $\partial\beta/\partial g > 0$ as this describes scaling at wavelengths which are long compared to the uv cut-off, $g_\bullet$ being an infra-red fixed point of the coupling constant flow $g(\mu)$. As shown in Appendix 3, the standard theory appertaining to this situation leads to the relation

$$G_R^{(2)}(s\mathbf{k}, \omega s^2, g_\bullet, \mu) = s^{2\eta_\nu^\bullet - \eta_\varepsilon^\bullet - d_0}\, G_R^{(2)}(\mathbf{k}, \omega s^{\eta_\nu^\bullet}, g_\bullet, \mu).$$

From this relation, we can infer the scaling of the energy spectrum $E(k) \sim k^{-\sigma}$ by integrating over the frequency to obtain

$$E(ks) = s^{-\sigma} E(k),$$

where

(5.3) $$\sigma = d_0 - d - 1 + \eta_\varepsilon^\bullet - \eta_\nu^\bullet.$$

We can eliminate η_ν from σ by making use of (4.5). Differentiating the latter, and using (4.1), we get

$$2\beta(g)g^{-1} + 4 = 3\eta_\nu^\bullet - \eta_\varepsilon^\bullet.$$

At the fixed point, since $\beta(g_\bullet) = 0$, this equation yields

$$\eta_\nu^\bullet = (4 + \eta_\varepsilon^\bullet)/3.$$

Substitution in (5.3) now gives

$$\sigma = d_0 - d - (7 - 2\eta_\varepsilon^\bullet)/3.$$

But we have seen in (2.14) that the theory is renormalisable in a dimension $d_0 = 4 + m = 7$. On the assumption that the results obtained from the renormalised theory can be continued down to the dimension $d = 3$ of physical space, the last equation yields

$$\sigma = \frac{5}{3} + \frac{2}{3}\eta_\varepsilon^\bullet,$$

which in view of (4.6) and (5.2) reduces to $\sigma = 5/3$ at 1-loop order.

A similar picture emerges in the case of structure functions. Consider, for example, the longitudinal structure functions

$$S_n(r) = \langle \Delta u_r^n \rangle,$$

where Δu_r is the difference of the x-components of **u** at two neighbouring points separated by a distance r : $\Delta u_r = u_1(x + r, y, z, t) - u_1(x, y, z, t)$.

Clearly, at 1-loop order the foregoing yields the scaling $S_2(r) \sim r^{2/3}$. So consider $S_4(r)$. Here one has to deal with correlation functions involving more than one field being evaluated at some point in space, such as $\langle u_1^2(x+r)u_1^2(x)\rangle$, which involves the technically difficult problem in field theory of composite operators (see e.g. Zinn-Justin, 1996, Ch. 12).

In general, a term of this sort introduces an additional renormalisation constant Z', say, because the coalescence of points induces new divergences in Fourier space, which cannot be accommodated by Z_ν and Z_ε. However, in the case of $S_4(r)$, one again finds, at 1-loop order, that $\Delta Z' = 0$. This, therefore, leads to the scaling

$$S_4(r) \sim r^{\varphi},$$

where

$$\varphi = 4 - 2\eta_\nu^\bullet.$$

Using (4.6) and (5.4), we get $\varphi = 4/3$, which again yields the K41 value.

Similarly, $S_6(r)$ involves new composite operators, such as $\langle u_1(x+r)^3 u_1(x)^3\rangle$. Because the composite operators embedded in structure functions change with order in this way one would not necessarily expect to find anomalous scaling exponents proportional to order. But whatever their form might take, it can only be determined at 2-loop order.

6. Discussion

The overall conclusion, therefore, is that only when the 2-loop approximation has been worked out in detail will we know whether useful quantitative results about intermittency can be derived from the Navier-Stokes equations using RG methods. So, what are the prospects for completing the 2-loop calculation of the scaling exponents? As regards the evaluation of ΔZ_ε the position is the following. As we have seen, ΔZ_ε can be found directly from the evaluation of $\Gamma^{(4)}$ in the static approximation, using the counter-term (4.4). At 2-loop order the diagrams contributing to $\Gamma^{(4)}$ have 30 different topologies. Whilst these exhibit uv divergence, extensive cancellations among diagrams occur, as in the case of 1-loop terms, with the result that one has to consider only the logarithmic divergences arising from the two new primitive diagrams, which are those shown in Figure 3. However, notwithstanding this considerable reduction of the magnitude of the task, the evaluation of the remaining diagrams is still a daunting undertaking, because of the complex form of the two vertices. Nevertheless, methods for handling such calculations do exist and it is hoped that their use will facilitate an eventual successful completion of the 2-loop approximation, by the present approach.

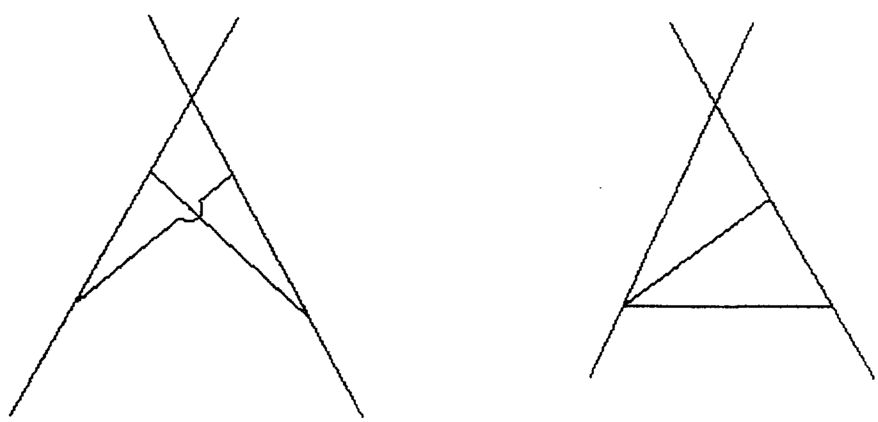

Fig. 3. – Primitive 2-loop diagrams for $\Gamma^{(4)}$.

Of course one is still working in low order perturbation theory even at 2-loop order, and the relevance of this approach to turbulence has often been questioned. For example, in a series of recent papers, L'vov and Procaccia (1995, 1996) have argued that intermittency is a non-perturbative effect. Similarly, Kraichnan (1982) argued, within the context of RG calculations, that intermittency is not treated adequately by low order perturbation theory. The present point of view is that we cannot yet tell whether such calculations really are useless or not until a full treatment of the 2-loop approximation had been completed. Clearly, if one found at 2-loop order that ΔZ_ε continues to vanish identically, as the results of L'vov and Procaccia would suggest, one's confidence in the method would be seriously undermined. However, the implication of the calculation of Teodorovich (1992) is that this might not be the case.

APPENDIX 1

Evalutation of the residues of the vertex functions

The term of (3.1) which contains the uv direction is readily identified by power counting to be $\Sigma(k)\, P_{\alpha\beta}(\mathbf{k})$ where

$$\Sigma(k) = \frac{1}{d-1} \int D\hat{\mathbf{q}}\, \frac{D(q)}{D(k)}\, U(\mathbf{k},\mathbf{q}) V(\mathbf{k},\mathbf{q}),$$

in which

$$U(\mathbf{k},\mathbf{q}) = P_{\alpha\beta\gamma}(\mathbf{k})\, P_{\alpha\gamma\delta}(\mathbf{k}-\mathbf{q})\, P_{\gamma\delta}(\mathbf{q}),$$

and

$$V(\mathbf{k},\mathbf{q}) = \int D\Omega \frac{1}{(\Omega-\omega)^2 + \nu^2|k-q|^4} \left\{ \frac{\nu k^2 - i\omega}{\nu q^2 + i\Omega} + \text{c.c.} \right\}.$$

The Ω integration is elementary and yields

$$V(\mathbf{k},\mathbf{q}) = \frac{1}{\nu q^2} \frac{\nu k^2 \{|k-q|^2 + q^2\} - \omega^2}{\nu q^2\, \nu^2\{|k-q|^2 + q^2\}^2 + \omega^2},$$

while evaluation of the projector product gives

$$U(\mathbf{k},\mathbf{q}) = k^2 \left\{ 1 - \left(\frac{k-q}{kq} \right)^2 \right\} \left\{ d - 1 - 2\frac{q^2\,(k^2 - \mathbf{k}.\mathbf{q})}{k^2 |\mathbf{k}-\mathbf{q}|^2} \right\}.$$

The last two results enable Σ to be written in terms of standard Feynmann integrals which can be evaluated in terms of the gamma function Γ (see Ramond (1989), Appendix B). After considerable manipulation, one finds that the divergent term is given by

$$\Sigma(k) = \frac{\alpha k^d}{4\nu} \Gamma\left(\frac{d_0 - d}{2} \right),$$

where

$$\alpha = \frac{d_0 - 1}{4(d_0+2)} \frac{S_{d_0}}{(2\pi)^{d_0}}.$$

Hence, the residue of $\Gamma^{(2)}_{\alpha\beta}$ at the pole $d = d_0$ is as stated.

Consider now the invariant $I_3(l) \equiv l_\beta \Gamma^{(3)}_{\alpha\beta\alpha}(\hat{l}, -\hat{l}, 0)$ in the static limit. Taking account of cancellation of terms between (3.2) and (3.3), and ignoring the convergent terms identifiable by power counting, we find that

$$I_3(l) = -\int D\mathbf{q}\, \frac{D(|\mathbf{l}-\mathbf{q}|)}{D(l)}\, l^2 P_{\alpha\sigma\nu}(\mathbf{l}) P_{\sigma\alpha\mu}(\mathbf{q})\, P_{\mu\nu}(\mathbf{l}-\mathbf{q})$$
$$\times \int D\Omega\, \frac{1}{(\Omega + i\nu q^2)\{\Omega^2 + \nu^2|\mathbf{l}-\mathbf{q}|^2\}}.$$

After performing the frequency integral and evaluating the projector product, we get

$$I_3(l) = \frac{i l^{d_0}}{2\nu^2} \int D\mathbf{q}\, \frac{1}{|\mathbf{l}-\mathbf{q}|^{d_0-2}[q^2 + |\mathbf{l}-\mathbf{q}|^2]}$$
$$\times \{d - 2 + (1 - 2\mathbf{q}.\mathbf{l}/q^2)(1 - 2\mathbf{q}.\mathbf{l}/l^2)\} \left\{1 - \left(\frac{\mathbf{l}(\mathbf{l}-\mathbf{q})}{l|\mathbf{l}-\mathbf{q}|}\right)^2\right\},$$

which can be expressed in terms of standard Feynman integrals, yielding a pole term

$$I_3(l) = \frac{i}{\nu} \frac{d_0 - 1}{d - d_0} \left(-\frac{\alpha l^{d_0}}{\nu}\right).$$

APPENDIX 2

Derivation of the scaling relation for $G_R^{(2)}$

Equation (5.1) is first solved using the method of characteristics. Let s be a parameter which measures distance along a characteristic. Along a characteristic the wavenumber scale μ, the frequency ω, and the coupling constant g become functions of s : $\mu(s)$, $\omega(s)$ and $g(s)$. We take $s = 1$ to be the point at which these functions assume the current values: $\mu(1) = \mu$, $\omega(1) = \omega$ and $g(1) = g$, and we choose s to correspond to a scale change by a factor s, so that $\mu(s) = \mu s$.

From (5.1) the change of g and ω along a characteristic is given by

$$s\frac{\mathrm{d}g}{\mathrm{d}s} = \beta(g) \qquad \text{and} \qquad s\frac{\mathrm{d}\omega}{\mathrm{d}s} = \omega\eta_\nu,$$

while the change in $G_R^{(2)}$ is given by

$$s\frac{\mathrm{d}G_R^{(2)}}{\mathrm{d}s} = (\eta_\varepsilon - 2\eta_\nu)\, G_R^{(2)}.$$

Integration of these equations with initial conditions corresponding to $s = 1$ yields

(A.1) $$G_R^{(2)}(k, \omega(s), g(s), \mu(s)) = G_R^{(2)}(k, \omega, g, \mu) \exp\left\{-\int_g^{g(s)} \frac{2\eta_\nu - \eta_\varepsilon}{\beta} \mathrm{d}g\right\}.$$

On the other hand standard dimensional analysis shows that $G_R^{(2)}$ has the general form

$$G_R^{(2)}(k, \omega, g, \mu) = \frac{\varepsilon}{\nu^2}\, \mu^{-d_0} F\left(\frac{k}{\mu}, \frac{\omega}{\nu\mu^2}, g\right),$$

from which it follows that

$$G_R^{(2)}(ks, \omega s^2, g, \mu s) = s^{-d_0} G_R^{(2)}(k, \omega, g, \mu).$$

Using this relation to eliminate $G_R^{(2)}$ from the right hand side of (A1) gives

$$G_R^{(2)}(ks, \omega s^2, g, \mu) = s^{-d_0} G_R^{(2)}(k, \omega(s), g(s), \mu) \exp\left\{ \int_g^{g(s)} \frac{2\eta_\nu - \eta_\varepsilon}{\beta} \mathrm{d}g \right\}. \tag{A2}$$

The scaling relation is obtained from the if limit of this equation corresponding to $s \to 0$. The derivation of this limit utilises the fact that the zero, $g_\bullet$, of $\beta(g) = 0$ is a fixed point of the coupling constant flow along the characteristic, because $\mathrm{d}g/\mathrm{d}s$ vanishes at this point. In the present case, in which $\mathrm{d}\beta/\mathrm{d}g > 0$ at this fixed point, $g_\bullet$ is a stable if fixed point as $s \to 0$. In these circumstances the integral in the exponential of (A2) is dominated by the region $g \sim g_\bullet$, and so the exponential yields a factor $s^{2\eta_\nu^\bullet - \eta_\varepsilon^\bullet}$. Similarly, $\omega(s)$ tends to $\omega s^{\eta_\nu^\bullet}$. Substitution of these results in (A2) now yields the scaling relation given in Section 5.

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